The author is grateful to S. I. Vyboronov for the discussion and streamline calculations.

LITERATURE CITED

1. Yu. V. Sanochkin, "Thermocapillary convection in a thin liquid layer heated locally from above," Zh. Prikl. Mekh. Tekh. Fiz., No. 6 (1983).
2. Yu. V. Sanochkin, "Steady-state thermocapillary motion in a horizontal liquid metal layer heated locally from above," Izv. Akad. Nauk SSSR, Mekh. Zhidk. Gaza, No. 6 (1984).
3. S. I. Vyboronov and Yu. V. Sanochkin, "Thermocapillary cell in a heavy liquid layer from above," Izv. Akad. Nauk SSSR, Mekh. Zhidk. Gaza, No. 1 (1985).
4. A. V. Zharinov and Yu. V. Sanochkin, "Possible explanation of the mechanism of cathode spot motion," Pis'ma Zh. Eksp. Teor. Fiz., 36, No. 5 (1982).

EFFECT OF SPATIAL MODULATION OF THE TEMPERATURE DISTRIBUTION ON THE STABILITY OF TWO-DIMENSIONAL STEADY FLOW IN A HORIZONTAL LAYER OF A TWO-COMPONENT LIQUID

V. A. Batishchev, V. V. Kolesov,
S. K. Slitinskaya, and V. I. Yudovich

UDC 536.25

We study the stability of two-dimensional steady flow in a horizontal layer of viscous heat-conducting liquid containing an admixture. For constant temperatures of the boundaries of the layer the convection equations admit a steady-state solution (mechanical equilibrium) which is stable if the temperature gradient is not too large. Under spatial modulation of the temperature distribution the liquid cannot be in equilibrium, and a spatially periodic convective regime is established in it for arbitrarily small temperature gradients [1, 2]. The purpose of the present article is to find the critical values of the temperature gradient for which this primary regime becomes unstable and a secondary regime develops in the liquid. A similar problem was solved in [2] for a homogeneous liquid when both boundaries of the layer are free surfaces.

1. Formulation of the Problem.

Suppose a viscous heat-conducting liquid containing an admixture fills an infinite plane horizontal layer of thickness h. The lower boundary of the layer is a solid surface whose temperature is modulated by small-amplitude perturbations which are periodic along the layer. The free upper surface of the layer is not deformed (taking account of the deformability is important only for thin layers of liquid and in weak gravitational fields [3]), and it is free of tangential stresses. The atmosphere above the layer is a stationary gas having a quasistationary temperature distribution. The heat flux Q along the vertical in the atmosphere far from the free surface is assumed given (for heating from below $Q > 0$). We assume that the temperature and the normal component of the heat flux are continuous through the free surface. There is no flow of the admixture through the boundaries of the layer. The liquid as a whole cannot be displaced parallel to the bottom. The amount of admixture in the liquid is specified.

The problem of determining the velocity $\mathbf{v} = \{v_x, v_y, v_z\}$, the pressure Π, the temperature T of the liquid, the temperature Θ of the atmosphere, and the concentration S of the admixture, reduced to dimensionless form and written in the Boussinesq approximation, has the form

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}, \nabla)\,\mathbf{v} = -\nabla\Pi + \Delta\mathbf{v} + \mathbf{e}\,(GT - G_s S),$$
$$\frac{\partial T}{\partial t} + (\mathbf{v}, \nabla)\,T = \frac{1}{\Pr}\Delta T, \quad \operatorname{div}\mathbf{v} = 0, \quad \Delta\Theta = 0, \qquad (1.1)$$
$$\frac{\partial S}{\partial t} + (\mathbf{v}, \nabla)\,S = \frac{1}{\Pr_d}\operatorname{div}(\nabla S + \xi S\nabla T),$$

Rostov-on-Don. Translated from Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, No. 2, pp. 83-86, March-April, 1985. Original article submitted February 13, 1984.

 0021-8944/85/2602-0226$09.50

$$\mathbf{v}=0,\quad T=\varepsilon\cos\omega x,\quad \frac{\partial S}{\partial z}+\xi S\frac{\partial T}{\partial z}=0\quad(z=0),$$

$$v_z=\frac{\partial v_x}{\partial z}+\frac{\partial v_z}{\partial x}=\frac{\partial v_y}{\partial z}+\frac{\partial v_z}{\partial y}=T-\Theta=\frac{\partial T}{\partial z}-m\frac{\partial\Theta}{\partial z}=\frac{\partial S}{\partial z}+\xi S\frac{\partial T}{\partial z}=0\quad(z=1),$$

$$\nabla\Theta\to\{0,\,0,\,-1/m\}\quad(z\to\infty),$$

where x, y, z are Cartesian coordinates with the x axis along the layer and the z axis perpendicular to the boundaries of the layer, t is the time, $\mathbf{e}=\{0,\ 0,\ 1\}$ is the unit vector along the z axis, $G=g\beta h^4Q/\varkappa\nu^2$ and $G_S=g\beta_S h^3 S/\nu^2$ are the Grashof number and its concentration analog, $Pr=\nu/\chi$ and $Pr_d=\nu/d$ are the Prandtl number and its diffusion analog, $\xi=khQ/\varkappa$ is a parameter characterizing the thermal diffusion, $m=\varkappa_0/\varkappa$ is the ratio of the thermal conductivities of the atmosphere $\varkappa_0$ and the liquid $\varkappa$, g is the acceleration due to gravity, $\bar{S}$ is the average concentration of the admixture under isothermal conditions; ν, χ, β, β_S, d and k are, respectively, the kinematic viscosity, the diffusivity, the coefficient of thermal expansion, the concentration compressibility, the diffusion coefficient of the admixture, and the thermal diffusion coefficient; ε and ω are the amplitude and frequency of modulation of the temperature on the lower boundary of the layer.

It is required to find the two-dimensional (independent of y) steady-state solution of problem (1.1) with a period $2\pi/\omega$ along the x axis (the primary steady-state regime), and to investigate its stability in the class of two-dimensional perturbations with two fundamental frequencies ω and α in x, where α is an arbitrary specified wave number.

2. Primary Steady-State Regime. If there is no modulation of the temperature distribution ($\varepsilon=0$), problem (1.1) admits the steady-state solution (mechanical equilibrium):

$$\mathbf{v}^0=0,\ T^0=-z,\ S^0=\xi\exp(\xi z)/[\exp(\xi)-1],$$
$$\Theta^0=\frac{1-m-z}{m},\quad \Pi^0=\int\limits_0^z(GT^0-G_SS^0)\,dz+\text{const}. \tag{2.1}$$

A small-amplitude modulation of the temperature transforms the one-dimensional solution (2.1) into a two-dimensional steady-state solution which, in the nonresonance case, we seek in the form of series in powers of the small parameter ε:

$$\begin{aligned}
v_{x0}&=\varepsilon u_{01}(z)\sin\omega x+\varepsilon^2u_{02}(z)\sin2\omega x+\ldots,\\
v_{z0}&=\varepsilon w_{01}(z)\cos\omega x+\varepsilon^2w_{02}(z)\cos2\omega x+\ldots,\\
\Pi_0&=\Pi^0+\varepsilon p_{01}(z)\cos\omega x+\varepsilon^2[p_{02}(z)\cos2\omega x+p_{00}(z)]+\ldots,\\
T_0&=T^0+\mathrm{Pr}\,\varepsilon\tau_{01}(z)\cos\omega x+\mathrm{Pr}\,\varepsilon^2[\tau_{02}(z)\cos2\omega x+\tau_{00}(z)]+\ldots,\\
\Theta_0&=\Theta^0+\mathrm{Pr}\,\varepsilon\theta_{01}(z)\cos\omega x+\mathrm{Pr}\,\varepsilon^2\theta_{02}(z)\cos2\omega x+\ldots,\\
S_0&=S^0+\xi\,\mathrm{Pr}_d\,\varepsilon s_{01}(z)\cos\omega x+\xi\,\mathrm{Pr}_d\varepsilon^2[s_{02}(z)\cos2\omega x+s_{00}(z)]+\ldots
\end{aligned} \tag{2.2}$$

Substituting (2.2) into (1.1), equating coefficients of equal powers of ε, and separating variables, we obtain a recurrent chain of linear boundary value problems for determining the coefficients in series (2.2). These boundary value problems are too cumbersome to write down here.

3. Stability of the Primary Steady-State Regime. The solution (2.2) of problem (1.1) exists for any value of the Rayleigh number R = PrG, but when the Rayleigh number passes through the critical value R_0, the solution (2.2) can become unstable. We seek R_0 in the form of the series

$$R_0=R_0^{(0)}+\varepsilon R_0^{(1)}+\varepsilon^2R_0^{(2)}+\ldots \tag{3.1}$$

To determine the coefficients in series (3.1) we impose infinitesimal two-dimensional perturbations on the primary regime (2.2), i.e., we seek a solution of problem (1.1), different from (2.2), in the form

$$v_x=v_{x0}+v_x',\quad v_z=v_{z0}+v_z',\quad \Pi=\Pi_0+\Pi',$$
$$T=T_0+\mathrm{Pr}\,T',\ \Theta=\Theta_0+\mathrm{Pr}\,\Theta',\ S=S_0+\xi\,\mathrm{Pr}_dS'. \tag{3.2}$$

Since problem (1.1) is invariant under the inversion $x\to-x$, $v_x\to-v_x$, the eigenfunctions (normal modes) of the linearized stability problem for regime (2.2) are divided into even and odd classes: v_z', Π', T', and Θ', and S' are even functions of x, and v_x' is odd; v_z', Π', T', Θ', and S' are odd functions of x, and v_x' is even.

Let us consider the case of even perturbations. Substituting (3.2) into (1.1), linearizing the problem obtained in the neighborhood of regime (2.2), and separating variables, we obtain

$$v_x' = u_1(z)\sin\alpha x + \varepsilon[u_{11}(z)\sin(\omega+\alpha)x + u_{12}(z)\sin(\omega-\alpha)x] + \\ + \varepsilon^2[u_{21}(z)\sin(2\omega+\alpha)x + u_{22}(z)\sin(2\omega-\alpha)x + u_{20}(z)\sin\alpha x] + \ldots,$$

$$v_z' = w_1(z)\cos\alpha x + \varepsilon[w_{11}(z)\cos(\omega+\alpha)x + w_{12}(z)\cos(\omega-\alpha)x] + \\ + \varepsilon^2[w_{21}(z)\cos(2\omega+\alpha)x + w_{22}(z)\cos(2\omega-\alpha)x + w_{20}(z)\cos\alpha x] + \ldots,$$

where α is the wave number of the perturbations ($\alpha \neq \omega$).

Analogous expansions in cosines hold for the functions Π', T', Θ', and S'. It is characteristic that the Fourier expansions of the perturbations contain only the harmonics $\sin(k\omega + \alpha)x$ and $\cos(k\omega + \alpha)x$ ($k = 0, 1, 2$) which arise in the interaction of the harmonics $\sin k\omega x$ and $\cos k\omega x$ of regime (2.2) with the fundamental harmonics $\sin\alpha x$ and $\cos\alpha x$ of the perturbations.

Substituting (3.2) into (1.1), linearizing the problem obtained in the neighborhood of the two-dimensional steady-state regime (2.2), equating coefficients of equal powers of ε, and separating variables, we obtain a recurrent chain of linear boundary value problems. The first of these problems serves to determine the leading term $R_0^{(0)}$ in the expansion of the critical value R_0 of the Rayleigh number in series (3.1) and the corresponding eigenfunctions. The subsequent boundary value problems are inhomogeneous. The conditions of their solvability enable us to find the remaining coefficients in the expansion (3.1).

Omitting the cumbersome calculations, we present only the results:

$$\begin{gathered}
R_0^{(1)} = 0, \quad R_0^{(2)} = I_2/2\alpha I_1, \quad I_1 = \int_0^1 (\mu s_1 - \tau_1)\, w_1 dz, \\
I_2 = \int_0^1 \left[f_1 w_1 + \alpha^2 R_0^{(0)} \Pr\left(1 + \mu\frac{\Pr}{\Pr_d}\right) f_2\tau_1 + \alpha^2 R_0^{(0)} \mu \Pr_d f_3\left(s_1 + \frac{\Pr}{\Pr_d}\tau_1\right)\right] dz, \\
f_1 = [(w_{12} - w_{11})(D^3 w_{01} + \alpha^2 D w_{01}) - (D^2 w_{12} - D^2 w_{11}) D w_{01}]/\omega \\
+ [w_{01} D^3 w_{11} - \alpha(\alpha + 2\omega) w_{01} D w_{11} - D^2 w_{01} D w_{11}]/(\alpha + \omega) + [w_{01} D^3 w_{12} - \\
- \alpha(\alpha - 2\omega) w_{01} D w_{12} - D^2 w_{01} D w_{12}]/(\alpha - \omega) - 2\alpha(w_{12} + w_{11}) D w_{01}, \\
f_2 = D w_{01}[(\alpha - \omega)\tau_{12} - (\alpha + \omega)\tau_{11}]/\omega - w_{01}(D\tau_{12} + D\tau_{11}) + \omega\tau_{01} \times \\
[D w_{12}/(\alpha - \omega) - D w_{11}(\alpha + \omega)] - D\tau_{01}(w_{12} + w_{11}) - 2 w_1 D\tau_{00}, \\
f_3 = D w_{01}[(\alpha - \omega) s_{12} - (\alpha + \omega) s_{11}]/\omega - w_{01}(D s_{12} + D s_{11}) + \omega s_{01} \times \\
[D w_{12}/(\alpha - \omega) - D w_{11}/(\alpha + \omega)] - D s_{01}(w_{12} + w_{11}) - 2 w_1 D s_{00}.
\end{gathered} \tag{3.3}$$

Here $D = d/dz$, and $\mu = \beta_S k\chi\bar{S}/\beta d$ is the thermoconcentration parameter.

In deriving (3.3) it was assumed that $\xi = khQ/\varkappa \approx 0$. Thereby we are limited to the consideration of a layer which is not too thick, and to liquids for which the thermal diffusion coefficient k is small in comparison with the thermal conductivity $\varkappa$. There is clearly no difficulty in considering the case $\xi \neq 0$, but then formulas (3.3) become much more cumbersome. For odd perturbations formulas (3.3) retain their form.

4. Numerical Results. To find the first two nonzero coefficients in the expansion of the critical value of the Rayleigh number in series (3.1) it is necessary to solve the spectral problem for the determination of $R_0^{(0)}$, w_1, τ_1, s_1, and three inhomogeneous boundary value problems for the determination of the functions w_{01}, τ_{01}, s_{01}, w_{11}, τ_{11}, s_{11}, w_{12}, τ_{12}, and s_{12}. Each of these problems was reduced to a boundary value problem for a system of eight ordinary differential equations of the first order, which were solved on a BÉSM-6 computer by the method of ranging.

Calculations were performed for $\Pr = 7$, $\Pr_d = 813$, $m = 0.0436$, and $\mu > 0$, which corresponds to a layer of sea water (the admixture in the water is salt) whose free surface is in contact with the air.

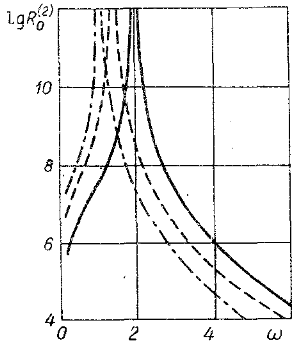

Fig. 1

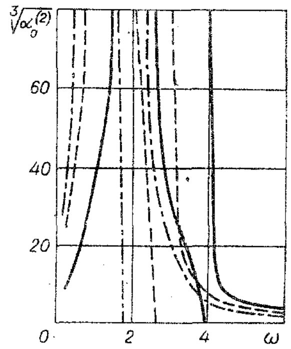

Fig. 2

Since the value α_0 of the wave number α of the perturbation which yields the minimum critical value of the Rayleight number is of the greatest physical interest, we performed a numerical minimization of $R_0(\alpha)$ with respect to α. It is easy to see that α_0 is decomposed into the series

$$\alpha_0 = \alpha_0^{(0)} + \varepsilon^2\alpha_0^{(2)} + \ldots,$$

where $\alpha_0^{(o)}$ is the value of the wave number α which minimizes the function $R_0^{(o)}(\alpha)$, and the correction $\alpha_0^{(2)}$ is given by the formula

$$\alpha_0^{(2)} = -\left[\frac{dR_0^{(2)}}{d\alpha}\Big/\frac{d^2R_0^{(0)}}{d\alpha^2}\right]_{\alpha=\alpha_0^{(0)}}.$$

The calculated dependences of the corrections $R_0^{(2)}$ and $\alpha_0^{(o)}$ to the critical values $R_0^{(o)}$ and $\alpha_0^{(o)}$ of the Rayleigh number and the wave number of the perturbations on the frequency ω of the modulation of the temperature distribution are shown in Figs. 1 and 2 for various values of the thermoconcentration parameter μ. The solid curves correspond to $\mu = 0.1$ ($\alpha_0^{(o)} = 2.00$, $R_0^{(o)} = 592$), the dashed curves to $\mu = 1$ ($\alpha_0^{(o)} = 1.45$, $R_0^{(o)} = 256$), and the dash-dot curves to $\alpha = 2$ ($\alpha_0^{(o)} = 1.09$, $R_0^{(o)} = 150$). The curves for $R_0^{(2)}(\omega)$ and $\alpha_0^{(2)}(\omega)$ have discontinuities at the points $\omega = \alpha_0^{(o)}$ and $\omega = 2\alpha_0^{(o)}$, since resonances occur in the interaction of the fundamental frequency with the harmonic temperature oscillations with frequencies ω near $\alpha_0^{(o)}$ and $2\alpha_0^{(o)}$. Expansion (3.1) no longer holds at these two exceptional points.

In conclusion, we note that for noncritical values of the wave number of the perturbations ($\alpha \neq \alpha_0^{(o)}$) the number of resonances is increased. If we drop terms of $O(\varepsilon^3)$ in expansions (2.2) and (3.3), the total number of resonances is no more than five.

LITERATURE CITED

1. G. Z. Gershuni and E. M. Zhukhovitskii, Convective Stability of an Incompressible Liquid [in Russian], Nauka, Moscow (1972).
2. L. P. Vozovoi and A. A. Nepomnyashchii, "Convection in a horizontal layer with spatial modulation of the temperature on the boundaries," in: Hydrodynamics No. 7 [in Russian], Perm' (1974).
3. V. Kh. Izakson and V. I. Yudovich, "Convection development in a liquid layer with a free boundary," Izv. Akad. Nauk SSSR, Mekh. Zhidk. Gaza, No. 4 (1968).